5th International Conference on Control Engineering&Information Technology (CEIT-2017)
Proceeding of Engineering and Technology –PET
Vol.33 pp. 19-22

Uncertain system's stabilization by an off-line Robust MPC based on polyhedral sets interpolation

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***Abstract*— In this paper, stabilization of uncertain systems using invariant polyhedral sets was proposed. The obtained full state feedback control laws are computed by an off-line approach reducing computational burdens. By minimizing a quadratic objective subject to linear constraints, the computed laws enhanced system performances. A comparison between implementing simple control laws and using an additional interpolation step will be carried out. To illustrate the effect of the computed laws, an example showed the enhancement of control performance.**

Keywordst— Robust Model Predictive Control, Polyhedral invariant set, Stabilization, Uncertain systems.

1- INTRODUCTION

Robust model predictive control (MPC) is an optimal technique for controlling systems subject to constraints on the states and on the inputs. This fact has proved very successful in the process industry and in academia. The synthesis approaches for on-line robust MPC have been studied by many researchers.

For linear systems, an on-line optimization is used to predict the process dynamics along a finite-time horizon [3]. Robust constrained MPC using linear matrix inequalities (LMI) was proposed by [7]. Brooms [1] presented a robust MPC for uncertain linear systems with ellipsoidal target sets, and Pornchai et al. [4], took the premise of a linear time varying process where uncertainty lay inside a polytope and arise a LMI computation at each sampling time to get the control law which is assumed to be linear in the predictions [5]. Wan and kothare [9] proposed an off-line robust constrained MPC algorithm. By choosing a sequence of states, converging to the origin, he constructed nested asymptotically stable invariant ellipsoids. Ellipsoidal approximations of the exact controllable sets are computed off-line and a numerically low demanding optimization problem is solved on-line.
Recently, polyhedral sets [2], [8] became wide-spread due to their flexibility of representation and reliable numerical algorithms. But computation of robust control invariant set is still a difficult problem for general nonlinear systems with bounded disturbances.
Polyhedral invariant sets can be imposed on a terminal state by means of linear constraints instead of quadratic constraints [6].

In this paper we show how to compute a sequence of state feedback control laws corresponding to a sequence of polyhedral invariant sets by solved in off-line step. On-line we select the smallest polyhedral invariant set which contains the current state at every time instant and get the local state feedback controller inside the polyhedral set. For better performance, we use an external input signal with the local controller and its resolution procedure followed by three types of interpolation is given. The efficiency of the considered external input signal and interpolations are then illustrated by an example.

This paper is organised as follows, a model with polytopic uncertainty is first presented. Then, the optimal control problem for constrained uncertain systems is formulated. Its resolution procedure using polyhedral invariant sets with an external input signal is added for the input control, is proposed. The efficiency of the proposed solution is then illustrated by an example. Finally, the paper is concluded.

2- MODEL WITH POLYTOPIC UNCERTAINTY

Considering the following linear time-varying (LTV) system with polytopic uncertainty:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) \\ y(k) = Cx(k) \end{cases} \quad (1)$$

ISSN 2356-5608

$$[A(k), B(k)] \in \Omega \quad (2)$$

$$|u_h(k+1/k)| \le u_{h,max},\ h = 1,2,\ldots,n_u \quad (3)$$

$$|y_r(k+1/k)| \le y_{r,max},\ r = 1,2,\ldots,n_y \quad (4)$$

where $x(x)$ is the state of the plant, $u(k)$ is the control input, *y(k)* is the plant output and Ω is the polytope

$$\Omega = conv\{[A_1, B_1], [A_2, B_2], \ldots, [A_L, B_L]\} \quad (5)$$

where $conv$ is the convex hull and Ω is a polytope, $[A_j, B_j]$ are vertices of the polytope such that:

$$[A_j, B_j] = \sum_{j=1}^{L} \lambda_j\, [A_j, B_j],\ \sum_{j=1}^{L} \lambda_j = 1, 0 \le \lambda_j \le 1, \quad (6)$$

The aim is this research is to find a state-feedback control law:

$$u(k+i\backslash k) = Kx(k+i\backslash k) + c(k+i\backslash k) \quad (7)$$

For $i = 1,\ldots,N-1$

that stabilizes (1) with the following performance cost:

$$\min_{u(k+i/k)} \max_{[A(k+i),B(k+i)]\in\Omega, i\ge 0} J_\infty(k)$$

$$J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix} \quad (8)$$

subject to (3) and (4).

where $\Theta > 0$ and $R > 0$ are symmetric weighting matrices.

In the work by Pornchai et al. [4] have used polyhedral invariant sets with an off-line robust algorithm to stabilize uncertain systems. We intend to use this algorithm with suitable interpolations and we add for the input control an external input signal to obtain better control performances.

3- Robust MPC Algorithm

This algorithm is based on two main steps. The first, solved off-line, allows obtaining a sequence of state feedback control laws corresponding to a sequence of polyhedral invariant sets. The second consists on an on-line implementation of state feedback control laws based on the position of the current state and an external input signal. Then, we consider two types of interpolation between two control laws corresponding to two adjacent polyhedral invariant sets.

3.1- OFFLINE STEP FOR CONSTRUCTION OF POLYHEDRL INVARIANT SET

Step 1: Choose a state sequence $x_i,\ i \in \{1, 2 \ldots, N\}$ and solve the following problem to obtain corresponding state feedback gains:

$$K_i = Y_i Q_i^{-1} \quad (9)$$

The states x_i must be chosen such that the distance between x_{i+1} and the origin is less than the distance between x_i and the origin.

Matrices Y_i and Q_i , for all $i = 1,2,\ldots,N$ are solutions of the following problem:

$$\min_{\gamma_i, Q_i, Y_i} \gamma_i \quad (10)$$

$$\text{subject to } \begin{bmatrix} 1 & x_i^T \\ x_i & Q_i \end{bmatrix} \ge 0 \quad (11)$$

$$\begin{bmatrix} Q_i & Q_i A_j^T + Y_i^T B_j^T & Q_i \Theta^{1/2} & Y_i^T R^{1/2} \\ A_j Q_i + B_j Y_i & Q_i & 0 & 0 \\ \Theta^{1/2} Q_i & 0 & \gamma_i I & 0 \\ R^{1/2} Y_i & 0 & 0 & \gamma_i I \end{bmatrix} \ge 0, \forall j = 1,2,\ldots,L \quad (12)$$

$$\begin{bmatrix} X & Y_i \\ Y_i^T & Q_i \end{bmatrix} \ge 0,\ X_{hh} \le u_{h,max}^2,\ h = 1,2,..,n_u \quad (13)$$

$$\begin{bmatrix} S & C(A_j Q_i + B_j Y_i) \\ (A_j Q_i + B_j Y_i)^T C^T & Q_i \end{bmatrix} \ge 0,$$

$$S_{rr} \le y_{r,max}^2, r = 1,2,\ldots,n_y, \forall j = 1,2,\ldots,L. \quad (14)$$

Step 2: Given the state feedback gains:

$$K_i = Y_i Q_i^{-1},\ i \in \{1, 2, \ldots, N\} \quad (15)$$

from step 1. For each K_i, the corresponding polyhedral invariant sets defined by:

$$S_i = \{x/M_i x \le d_i\} \quad (16)$$

are constructed by the following :

- **Step 2.1:** Set $M_i = [C^T, -C^T, K_i^T, -K_i^T]$, $d_i = [y_{max}^T, y_{min}^T, u_{max}^T, u_{min}^T]^T$ and $m = 1$.

- **Step 2.2 :** Select row m from $(M_i,\ d_i)$ and check whether $M_{i,m}(A_j + B_j K_i)x \le d_i$ is redundant with respect to the constraints defined by $(M_i,\ d_i)$ by solving the problem:

$$\max_x\ W_{i,m,j}$$

$$\text{subject to } W_{i,m,j} = M_{i,m}(A_j + B_j K_i)x - d_{i,m}, \quad (17)$$

$$M_i x \le d_i$$

- **Step 2.3:** Let $m = m + 1$ and return to Step 2.2. If m is strictly larger than the number of rows in (M_i, d_i) then terminate.

3.2- ON-LINE STEP

- ***Calculate c(k)***

Solve the following QP problem at time *k*, we can get the value of
$c(k) = \{c(k\backslash k), \dots, c(k+N-1\backslash k)\}$. (18)

$$\min_{\substack{c(k)\\ \bar{x}(k+1),\dots,\bar{x}(k+N)\\ \underline{x}(k+1),\dots,\underline{x}(k+N)}} \sum_{l=1}^{N} c(k+l\backslash k)' c(k+l\backslash k) \tag{19}$$

With the following constraints:

$$\gamma_j^{+}\underline{x}(k+l) - \gamma_j^{-}\bar{x}(k+l) + Bc(k+l\backslash k) \geq \underline{x}(k+l+1)$$
$$j = 1, \dots, L, \quad l = 0, \dots, N-1 \tag{20}$$

$$\gamma_j^{+}\bar{x}(k+l) - \gamma_j^{-}\underline{x}(k+l) + Bc(k+l\backslash k) \leq \bar{x}(k+l+1)$$
$$j = 1, \dots, L, l = 0, \dots, N-1 \tag{21}$$

$$\varphi_i^{+}\bar{x}(k+l) - \varphi_i^{-}\underline{x}(k+l) + c(k+l\backslash k) \leq \bar{u} \tag{22}$$

$$\varphi_i^{+}\underline{x}(k+l) - \varphi_i^{-}\bar{x}(k+l) + c(k+l\backslash k) \geq \bar{u} \tag{23}$$

With

$$\bar{x}(k) = \underline{x}(k) = x(k),\ \varphi_i^{+} = \max(K_i, 0)$$
$$\varphi_i^{-} = \max(-K_i, 0),$$
$$\gamma_j^{+} = \max((A_j + BK_i), 0)$$
$$\gamma_j^{-} = \max(-(A_j + BK_i), 0)$$

- ***On-line Step without interpolation***

At each sampling time, determine the smallest polyhedral invariant set $S_i = \{x / M_i x \leq d_i\}$, $i = 1,2, \dots, N-1$ containing the measured states and implement the corresponding state feedback control law $u(k/k) = K_i x(k/k) + c(k)$ to the process.

- ***On-line Step with 2-points interpolation***

At each sampling time, if the measured state lies between S_i and S_{i+1}, $i = 1,2, \dots, N-1$ implement the interpolated gain obtained by :

$$K = \alpha K_i + (1-\alpha) K_{i+1} \tag{24}$$

where $0 < \alpha < 1$, K_i is the corresponding gain of the polyhedral invariant set S_i, $i = 1,2, \dots, N-1$.

- ***On-line Step with 3-points interpolation***

At each sampling time, if the measured state lies between S_i, S_{i-1} and S_{i-2}, implement the interpolated gain obtained by :

$$K = \alpha_1 K_{i-2} + \alpha_2 K_{i-1} + \alpha_3 K_i \tag{25}$$

where $0 < \alpha_i < 1$, for all $i = 1,2,3$ and $\sum_{i=1}^{3} \alpha_i = 1$.

4 – APPLICATION

We will consider the application of the proposed approach to an angular positioning system [7]. The system consists of an electric motor driving a rotating antenna so that it always points in the direction of a moving object. The motion of the antenna can be described by the following discrete time equation

$$\begin{bmatrix}\theta(k+1)\\ \dot{\theta}(k+1)\end{bmatrix} = \begin{bmatrix}1 & 0.1\\ 0 & 1-0.1\alpha(k)\end{bmatrix}\begin{bmatrix}\theta(k)\\ \dot{\theta}(k)\end{bmatrix} + \begin{bmatrix}0\\ 0.0787\end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\theta(k)\\ \dot{\theta}(k)\end{bmatrix} \tag{26}$$

where $\theta(k)$ is the angular position of the antenna, $\dot{\theta}(k)$ is the angular velocity of the antenna and $u(k)$ id the input voltage to the motor.

The uncertain parameter $\alpha(k)$ is proportional to the coefficient of viscous friction in the rotating parts of the antenna. It is assumed to be arbitrarily time varying in the range of

$$0.1 \leq \alpha(k) \leq 10. \tag{27}$$

Let $\bar{\theta} = \theta - \theta_{eq}, \bar{\dot{\theta}} = \dot{\theta} - \dot{\theta}_{eq}$ and $\bar{u} = u - u_{eq}$ where the subscript eq is used to denote the corresponding variable at equilibrium condition.

The discrete time model (26) can be written as follows

$$\begin{bmatrix}\bar{\theta}(k+1)\\ \bar{\dot{\theta}}(k+1)\end{bmatrix} = \begin{bmatrix}1 & 0.1\\ 0 & 1-0.1\alpha(k)\end{bmatrix}\begin{bmatrix}\bar{\theta}(k)\\ \bar{\dot{\theta}}(k)\end{bmatrix} + \begin{bmatrix}0\\ 0.0787\end{bmatrix} \bar{u}(k)$$

$$\bar{y}(k) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\theta(k)\\ \dot{\theta}(k)\end{bmatrix} \tag{28}$$

Because the uncertain parameter $\alpha(k)$ is varied between 0.1 and 10, we conclude that $A(k) \in \Omega$ where Ω is given as follows

$$\Omega = Co\left\{\begin{bmatrix}1 & 0.1\\ 0 & 0.99\end{bmatrix}, \begin{bmatrix}1 & 0.1\\ 0 & 0\end{bmatrix}\right\} \tag{29}$$

The objective is to regulate $\bar{\theta}$ from 0.2 to the origin by manipulating $\bar{u}$. The input constraint is $|\bar{u}(k)| \leq 2$ volts. Here $J_\infty(k)$ is given by (8) with $\Theta = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $R = 0.00002$.

Next, let's choose a sequence of states:

$$x_i = \left\{\begin{matrix}(0.35,0.35), (0.3,0.3),\\ (0.25,0.25), (0.02,0.2),\\ (0.15,0.15), (0.1,0.1)(0.05,0.05)\end{matrix}\right\} \tag{30}$$

is chosen to calculate the corresponding state feedback gains K_i.

This sequence is used to calculate seven state feedback gains K_i corresponding to seven polyhedral invariant sets (Figure 1).

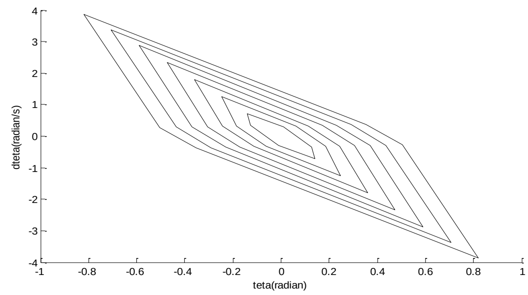

Figure 1: The obtained sequence of 7 polyhedral invariant sets constructed off-line

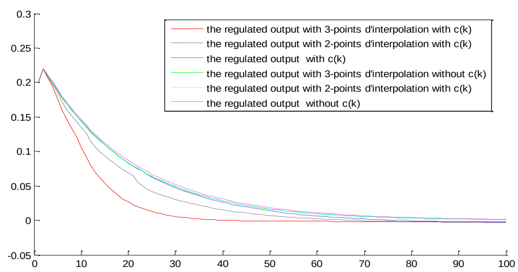

Figure 2: The regulated output

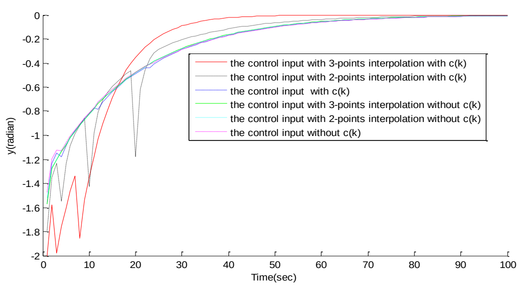

Figure 3: The control input

The closed-loop responses of the system when $\alpha(k)$ is randomly time-varying between $0.1 \leq \alpha(k) \leq 10$, are showed respectively by Figure 2 and Figure 3. It is seen that the considered interpolations, by 2- points and 3-points give less conservative results as compared to the approach without interpolation.

And we can observe that by using interpolation with external input signal $c(k)$ does provide better control performance.

5 - CONCLUSION

A constrained MPC algorithm based on polyhedral sets interpolation for polytopic uncertain systems is presented in this paper. An interpolation step applied to the obtained control laws based on polyhedral invariant sets gives good performance with the use of external perturbation input signal. Numerical example confirms that this method is effective and gives good performance.

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